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Mirror symmetry and toric geometry in three dimensional gauge theories

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Abstract: We study three dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry. We show that the Coulomb branches of such theories may be rendered compact by the dynamical generation of Chern-Simons terms and present a new class of mirror symmetric theories in which both Coulomb and Higgs branches have a natural description in terms of toric geometry.

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Contents

1. Introduction

Mirror symmetry of three dimensional gauge theories is an infra-red equivalence of two theories in which Coulomb and Higgs branches, and Fayet-Iliopoulos parameters and masses are exchanged. First discovered in $\mathcal{N} = 4$ theories by Intriligator and Seiberg [1], explanations in terms of string theory dualities [2] as well as generalisations to both other gauge groups [3] and $\mathcal{N} = 2$ gauge theories [4, 5] soon appeared. More recently it has been shown that, for a large class of three-dimensional theories, the correspondence may be extended to all length scales [6].

In this paper we present further examples of mirror theories with $\mathcal{N} = 2$ supersymmetry. A novel feature of these theories is that the Coulomb branch may be compact and is naturally described in the language of toric geometry. In fact we will find that the Coulomb branch, which in three dimensions admits a toric action, possesses submanifolds on which certain cycles of the torus vanish and thereby defines a toric variety. Further we find pairs of theories for which the Coulomb (Higgs) branch of one theory and the Higgs (Coulomb) branch of the other are specified by the same toric data. A classical analysis of the Higgs branch in question simply yields the usual symplectic quotient construction of the corresponding toric variety. In contrast, various quantum effects which are characteristic to three dimensions play a central role in realizing the same toric variety as the Coulomb branch of the mirror theory. This equivalence means, in particular, that the two branches admit identical U(1) isometries with precisely the same set of fixed submanifolds. As we are dealing with theories with only four supercharges it is not obvious whether the correspondence extends to the respective metrics on the two branches. Despite this, we will find that the two metrics do in fact agree in cases where an explicit calculation is possible.

The plan of the paper is as follows: in section 2 we review various aspects of abelian gauge theories in three dimensions. In section 3 we present a simple self-mirror theory for which both the Higgs and Coulomb branch is a copy if $\mathbb{CP}^1$. Finally, in section 4 we consider a more general abelian gauge theory and employ the language of toric geometry. The appendix contains a discussion of the brane realisation of the theory of section 3. As this work was in preparation, [7] appeared which considers similar theories. The connection between three-dimensional Chern-Simons gauge theories and the related phenomenon of mirror symmetry in two-dimensional Calabi-Yau sigma-models has been discussed in [8]

2. Review of $N = 2$ gauge theories

Three dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry (4 supercharges) have been studied in detail in [5], where a comprehensive introduction may be found. Here we collate some facts relevant to abelian gauge theories, with a $\mathrm{U}(1)^r$ gauge group and N chiral multiplets. To this end, we introduce abelian vector superfields V_a, $a = 1, \ldots, r$, and chiral superfields Q_i, $i = 1, \ldots, N$, both of which are simply the dimensional reduction of the familiar four dimensional $\mathcal{N} = 1$ superfields. The gauge kinetic terms are written most simply in terms of linear superfields $\Sigma_a = \epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta V_a$, whose lowest component is a real scalar ϕ_a and also includes the U(1) field strength as well as two Majorana fermions. The chiral multiplets each consist of a complex scalar q_i and two further Majorana fermions. The kinetic terms for all fields are written as D-terms,

$$\mathcal{L}_K = \int \mathrm{d}^4\theta \left[\sum_{i=1}^{N} Q_i^\dagger \exp\left(2\sum_{a=1}^{r} R_i^a V_a\right) Q_i + \sum_{a=1}^{r} \frac{1}{e_a^2}\Sigma_a^2 \right], \tag{2.1}$$

where e^a is the a^{th} gauge coupling constant which has dimension $(\mathrm{mass})^{1/2}$ and R_i^a are the charges of the chiral multiplets under each of the gauge symmetries. We assume these charges to be integers. Further interactions take the form of a superpotential, $\mathcal{W}$, constructed from gauge invariant monomials of the chiral superfields,

$$\mathcal{L}_F = \int \mathrm{d}^2\theta\, \mathcal{W}(Q_i) + \mathrm{h.c.}$$

In particular, if there exist two chiral superfields of opposite charge the usual complex mass is written is this manner. In three dimensions each chiral multiplet may have a further, real, mass parameter which cannot be written in terms of a superpotential and which will play an important role in the following discussion. It is introduced by weakly gauging the Cartan-subalgebra of the global flavour symmetry of the theory and constraining the vector multiplet scalar to a fixed background vacuum expectation value (VEV). The net result is that the exponent in (2.1) is replaced by,

$$\sum_{a=1}^{r} R_i^a V_a \to \sum_{a=1}^{r} R_i^a V_a + 2m_i\theta\theta^\dagger\,.$$

Notice that there are only $N-r$ independent such parameters, the remaining r being set to zero by shifts of the vector multiplet scalars ϕ_a.

Two further sets of couplings will also prove important in the story: Fayet-Iliopoulis (FI) parameters ζ_a which have dimension of mass, and dimensionless Chern-Simons (CS) parameters κ_{ab}. The former are incorporated in the usual fashion,

$$\mathcal{L}_{FI} = \sum_{a=1}^{r} \zeta_a \int \mathrm{d}^4\theta\, V_a\,,$$

while the latter are written in terms of the linear superfield as,

$$\mathcal{L}_{CS} = \sum_{a,b=1}^{r} \kappa_{ab} \int \mathrm{d}^4\theta\, \Sigma_a V_b\,,$$

Notice from the similarity of these two expression that the combinations of scalar fields $\sum_b \kappa_{ab}\phi_b$ will play the role of a dynamical FI parameter in the theory. This may be seen by examining the classical scalar potential, obtained by integrating out auxiliary fields,

$$U = \sum_{a=1}^{r} e_a^2 \left(\sum_{i=1}^{N} R_i^a |q_i|^2 - \sum_{b=1}^{r} \phi_b \kappa_{ab} - \zeta_a \right)^2 + \sum_{i=1}^{N} M_i^2 |q_i|^2 + \sum_{i=1}^{N} \left| \frac{\partial \mathcal{W}}{\partial q_i} \right|^2, \tag{2.2}$$

where

$$M_i = \sum_{a=1}^{r} R_i^a \phi_a + m_i \tag{2.3}$$

is the effective mass of the i^{th} chiral multiplet. The manifold of classical supersymmetric vacua, determined by the condition $U=0$, depends on the parameters ζ_a, κ_{ab} and m_i. Let us consider situations with $\mathcal{W}=0$. Then there may be two branches of vacua: the Higgs and Coulomb branches. In the former, the vector multiplet scalars are set to zero, $\phi_a = 0$, while the q_i's are constrained by the vanishing of the first expression in (2.2) modulo gauge transformations. In this phase the gauge symmetry is generically completely broken. It is clear from the form of (2.2) that the Higgs branch does not exist for generic non-zero real masses.

On the Coulomb branch, all chiral multiplet scalars are set to zero, $q_i = 0$, while the ϕ_a's are unconstrained. Again, from (2.2), it is clear that non-zero FI or CS parameters will lift the Coulomb branch. When this phase exists however the gauge symmetry is completely unbroken and one may exchange each abelian gauge field for a scalar, σ_a, of period e_a, via the duality transformation $F_a^{\mu\nu} = \epsilon^{\mu\nu\rho}\partial_\rho\sigma_a$. The Coulomb branch is therefore parametrised by the VEVs of both ϕ_a and σ_a, which combine to lie in r chiral multiplets, and is classically given by $(\mathbb{R}\times S^1)^r$. There exist r $\mathrm{U}(1)_J$ isometries of the Coulomb branch induced by constant shifts of the r dual photons. These are preserved in the full quantum theory [5]. In the following it will sometimes be useful to weakly gauge these symmetries. In particular, consider gauging the symmetry which shifts σ_a by a constant and leaves the other dual photons invariant. As explained in [5], the lowest component of the linear multiplet which contains the corresponding field strength is precisely the FI parameter ζ_a. In addition to pure Higgs and Coulomb branches there will also exist mixed branches of vacua in which both q's and ϕ's have non-zero VEV.

This concludes our discussion of the classical field theory. Let us now mention some relevant quantum effects that occur. Firstly note that in four dimensions a theory with the above matter content would suffer a gauge anomaly unless $\sum_i R_i^a = 0$ for each gauge group a. In three dimensions, while there are no gauge anomalies, the theory may suffer from a "parity anomaly" [9]. This manifests itself in the dynamical generation of CS terms due to integrating out chiral multiplet fermions at one-loop [5]. In the case of $\mathcal{W} = 0$, the effective CS term is given by,

$$\kappa_{ab}^{\text{eff}} = \kappa_{ab} + \frac{1}{2}\sum_{i=1}^{N} R_i^a R_i^b \,\text{sign}\, M_i\,. \tag{2.4}$$

The parity anomaly arises from the observation that gauge invariance requires κ_{ab}^{eff} to be an integer and therefore, for certain matter content, the factor of 1/2 in (2.4) means a non-zero bare CS term is obligatory, breaking parity.

A similar CS coupling is also generated for weakly gauged global symmetries of the type discussed earlier in this section. In particular, for the element of the Cartan-subalgebra of the global flavour group under which q_i transforms with charge $+1$, with all other fields neutral, one has

$$\kappa_{ai}^{\text{eff}} = \frac{1}{2} R_a^i \,\text{sign}\, M_i\,.$$

Finally, the combined effect of all dynamically generated CS parameters may be interpreted as a finite renormalisation of the FI parameters,

$$\zeta_a^{\text{eff}} = \zeta_a + \sum_{b=1}^{r} \kappa_{ab}^{\text{eff}}\phi_b + \sum_{i=1}^{N} \kappa_{ai}^{\text{eff}} m_i\,. \tag{2.5}$$

Notice that the gauge and global symmetries appear on an equal footing in this expression. Further, for $M_i = 0$, which is the case on the Higgs branch, $\text{sign}\, M_i$ and κ_{ab}^{eff} are ill-defined. However, the FI parameters ζ_a^{eff} are of the form $M_i \,\text{sign}\, M_i$ and are therefore continuous at $M_i = 0$. Finally, these one-loop corrections can be implemented in the scalar potential (2.2), simply by replacing the FI parameters and CS terms by the renormalised FI parameters (2.5).

There is one further quantum effect, discussed in [5], that will be important for us. At the intersection of Coulomb and Higgs branches, certain $\text{U}(1)_J$ isometries, which are generically broken on the Coulomb branch, must be restored as the Higgs branch is invariant under such symmetries. This, in turn, requires the vanishing of the periods of the dualized scalars at these points and the Coulomb branch is to be viewed as a fibre of $\mathbb{T}^r$ over R^r where certain cycles of the torus shrink at the intersection points. This suggests that the Coulomb branch can be thought of as a toric variety specified in terms of toric data which encodes its intersections with other branches. In the following Sections we will realize this idea explicitly.

3. A self-mirror example

In this section we exhibit a simple $\mathcal{N}=2$ gauge theory in which both Higgs and Coulomb branches are compact, the former classically and the latter due to the dynamical generation of CS terms. This will also serve as an pedagogical example for more complicated models to be discussed in the following section. The theory has a single abelian gauge group with bare FI parameter ζ. The matter content consists of two chiral multiplets, both transforming with charge +1 and with real masses $m_1 = -m_2$. We set both superpotential and bare CS parameter to zero so that the effective one-loop couplings are given by,

$$\kappa^{\rm eff} = \frac{1}{2}\left[\mathrm{sign}(\phi+m)+\mathrm{sign}(\phi-m)\right]. \tag{3.1}$$

Similarly, the one-loop effective FI parameter is,

$$\zeta^{\rm eff} = \zeta + \frac{1}{2}(\phi+m)\ \mathrm{sign}(\phi+m) + \frac{1}{2}(\phi-m)\,\mathrm{sign}(\phi-m) \tag{3.2}$$

and the scalar potential (2.2) thus simplifies to,

$$U = e^2\left(|q_1|^2+|q_2|^2-\zeta^{\rm eff}\right)^2 + (\phi+m)^2|q_1|^2 + (\phi-m)^2|q_2|^2\,.$$

Let us now examine solutions of the vacuum condition $U=0$ of this theory as a function of the FI and mass parameters. Without loss of generality we may restrict our attention to masses $m \geq 0$. We will also set the bare CS coupling to zero. There are then five different regimes:

i) $m=\zeta=0$: In this case there is a unique vacuum at the origin $\phi=q_1=q_2=0$.

ii) $m=0$, $\zeta>0$: In this case we have a Higgs branch of vacua given by $\phi=0$, and $|q_1|^2+|q_2|^2=\zeta$ modulo gauge transformations. This is simply Witten's gauged linear sigma model [10] with target space $\mathbb{CP}^1$ of Kähler class ζ.

iii) $m>0$, $\zeta=-m$: In this case, requiring $\zeta^{\rm eff}=0$ restricts the range of ϕ to $|\phi|\leq m$. Therefore the space of vacua is given by $q_1=q_2=0$ while ϕ may take any value in the interval, $I=\{\phi : -m\leq\phi\leq m\}$. We will discuss this case in more detail below.

iv) $m > 0$, $\zeta > -m$: In this case there are two isolated Higgs branch vacua. These are located at $\phi = -m$, $|q_1|^2 = m$, $q_2 = 0$ and $\phi = m$, $|q_2|^2 = m$, $q_1 = 0$ respectively.

v) $m \geq 0$, $\zeta < -m$: In this case there are two isolated vacua on the Coulomb branch located at $q_1 = q_2 = 0$, $\phi = \pm\zeta$.

The above vacuum structure may be reproduced by realising the theory as a D3-brane probe of a (p, q) 5-brane web. This is done in the appendix, where we also discuss the vacuum structure in the presence of a non-zero bare CS term.

The novel feature of case iii) above is the restriction of the range of ϕ to an interval I. For the critical value of the bare FI parameter, $\zeta = -m$, the Coulomb branch is therefore compact. After dualizing the massless photon in favour of a periodic scalar σ, the Coulomb branch can be thought as a fibration of S^1 over I. The $\mathrm{U}(1)_J$ symmetry which shifts σ by a constant acts on the fibre over each point. Classically the fibration is trivial and simply corresponds to the cylinder $I \times S^1$. However we will now argue that this picture is modified by quantum effects. The key idea, based on the arguments of [5], is that the endpoints of the interval, $\phi = \pm m$, must be invariant under $\mathrm{U}(1)_J$. This follows because, at these two points, the Coulomb branch intersects the Higgs branch of case iv) which must be invariant under $\mathrm{U}(1)_J$. Strictly speaking, moving onto the Higgs branch requires changing the bare FI parameter ζ away from its critical value. This means that one is moving to a different theory rather than onto another vacuum branch of the same theory. This distinction is irrelevant here because, as explained in the previous section, we can promote ζ to be the scalar component of a background linear multiplet by weakly gauging $\mathrm{U}(1)_J$. The existence of a new branch emanating from the endpoints of the interval then hinges on whether or not this field is massless. At a generic point on the Coulomb branch $\mathrm{U}(1)_J$ acts non-trivially on the dual photon is therefore spontaneously broken. This means that the corresponding gauge multiplet which contains ζ acquires a mass by the Higgs mechanism. Conversely, ζ only remains massless at points on the Coulomb branch at which $\mathrm{U}(1)_J$ remains unbroken.

The restoration of $\mathrm{U}(1)_J$ discussed above is only possible if the S^1 fibre shrinks to zero size at the endpoints of the interval. Following [5], this effect which can be ascribed to quantum corrections which are unsupressed near these points. The result is a Coulomb branch with the topology of a two sphere. As mentioned above, it is natural to combine ϕ and σ to form a complex scalar which is the lowest component of a chiral superfield. The Coulomb branch can then be thought of as a Kähler manifold of complex dimension one. We therefore seek such a complex manifold which can be realized as an S^1 fibration over an interval with degenerate fibres at the endpoints. This is precisely the data which specifies $\mathbb{CP}^1$ as a toric variety. In fact, as explained in [11], the interval is just the toric diagram for $\mathbb{CP}^1$. In the following we will see that this connection between three-dimensional Coulomb branches and toric diagrams is a general one.

In this simple case, one can also understand the symmetry between Higgs and Coulomb branches by considering the original $\mathcal{N}=4$ self-mirror theory of Intriligator and Seiberg [1]. In the $\mathcal{N}=2$ language this consists of a U(1) vector multiplet, a single neutral chiral multiplet, two chiral multiplets of charge +1 and two of charge −1. A superpotential couples all hypermultiplets. One may flow to the theory described above by gauging various global symmetries (including a subgroup of one the SU(2) R-symmetries of the $\mathcal{N}=4$ algebra) and introducing mass terms for the unwanted chiral multiplets. The duality properties should be invariant under such deformations [5], and the Higgs and Coulomb branches should again be equal in the above theory.

Finally, one may also see the emergence of the SU(2)-invariant Kähler metric on $\mathbb{CP}^1$ by an explicit one-loop calculation on the Coulomb branch. Note however that, unlike the theory with $\mathcal{N}=4$ supersymmetry, one has little control over the Kähler potential and the following calculation is only valid at points where $e^2 \ll |\phi \pm m|$ and, in particular, cannot be trusted at the end points of the interval. Nevertheless we proceed with the calculation, encouraged by the end result. Classically, the low-energy dynamics on the Coulomb branch are described by the metric,

$$\mathrm{d}s^2 = H(\phi)\mathrm{d}\phi^2 + H(\phi)^{-1}\mathrm{d}\sigma^2\,, \tag{3.3}$$

where classically $H = 1/e^2$. As well as the renormalisation of the FI and CS parameters discussed above, there is also a finite renormalisation of the gauge coupling constant,

$$\begin{aligned} H^{1-\mathrm{loop}} &= \frac{1}{e^2} + \frac{1/2}{|\phi+m|} + \frac{1/2}{|\phi-m|} \\ &= \frac{1}{e^2} + \frac{m}{m^2-\phi^2} \end{aligned}$$

where the second equality follows only when ϕ is restricted to the interval I. In the limit, $e^2 \to \infty$, we thus find that one-loop Coulomb branch metric indeed becomes the Fubini-Study metric on $\mathbb{CP}^1$ with Kähler class m. Notice that, as in the original Intriligator-Seiberg model, the $\mathrm{U}(1)_J$ symmetry of the Coulomb branch is enhanced in the infra-red to a SU(2) symmetry. Under mirror symmetry this is exchanged with the SU(2) flavour symmetry of the theory while the FI parameter is exchanged with the real mass.

To summarise, this model is self-mirror; the Coulomb and Higgs branches coincide if one simultaeously exchanges the mass and FI parameters. Specifically, the Coulomb branch exists only when $\zeta = 0$ and is given by $\mathbb{CP}^1$ of Kahler class m. In constrast, the Higgs branch only exists when $m = 0$ and is given by $\mathbb{CP}^1$ of Kahler class ζ.

4. Mirror symmetry and toric geometry

In this section we discuss the more general mirror symmetric theories. Specifically, we will consider,

Theory A: $\mathrm{U}(1)^r$ gauge theory with N chiral multiplets containing scalars q_i of charge R_i^a, $i = 1, \ldots, N$ and $a = 1, \ldots, r$. The parameters of the model are bare CS couplings κ_{ab}, bare FI parameters ζ_a and real masses m_i. Notice that only $N - r$ of the mass parameters are independent.

Theory B: $\mathrm{U}(1)^{N-r}$ gauge theory with N chiral multiplets containing scalars $\tilde{q}_i$ of charge S_i^u, $u = 1, \ldots, N-r$. The parameters are $\tilde{\kappa}_{uv}$, $\tilde{\zeta}_u$ and $\tilde{m}_i$, where $N-(N-r) = r$ of the mass parameters are independent.

The charges of theory A and theory B are constrained to satisfy,

$$\sum_{i=1}^{N} R_i^a S_i^u = 0 \qquad \text{for all } a \text{ and } u\,. \tag{4.1}$$

We denote the Coulomb branch of theory A (B) as $\mathcal{M}_C^{A(B)}$ and the Higgs branch as $\mathcal{M}_H^{A(B)}$. Notice firstly that $\dim(\mathcal{M}_C^A) = \dim(\mathcal{M}_H^B) = r$ and $\dim(\mathcal{M}_C^B) = \dim(\mathcal{M}_H^A) = N - r$. Similarly, the number of independent mass parameters of theory A (B) is equal to the number of FI parameters of theory B (A).

Let us start with the Higgs branch of theory B in the case where the masses and CS parameters vanish. This is defined as the symplectic quotient,

$$\mathcal{M}_H^B = \mu_u^{-1}(\tilde{\zeta}_u)/\,\mathrm{U}(1)^{N-r}\,,$$

where the momentum map is $\mu_u = \sum_i S_i^u |\tilde{q}_i|^2$. For certain values of the FI parameters (specifically, when ζ_a lie in the Kähler cone of $\mathcal{M}_H^B$ — see [12]), the classical Higgs branch is the toric variety,[1]

$$\mathcal{M}_H^B = (\mathbf{C}^N - F_\Delta)/(\mathbf{C}^\star)^{N-r}$$

in which $\mathbf{C}^N$ is parametrised by $\tilde{q}_i$, and F_Δ is a subset of $\mathbf{C}^N - \mathbf{C}^{\star N}$. Precisely which subset is determined in terms of a fan of cones, Δ, as will be reviewed below. The action of $(\mathbf{C}^\star)^{N-r}$ in the above quotient is the complexified gauge symmetry of theory B, given by,

$$\tilde{q}_i \to \lambda^{S_i^u}\tilde{q}_i \qquad \lambda \in \mathbf{C}^\star\,,\; u = 1, \cdots N-r\,.$$

Our first task is to identify the set F_Δ. This is specified by the charges S_i^u, which we may view as $(N-r)$ charge vectors in $\mathbb{Z}^N$. The first step is to construct r vectors

[1] For an introduction to toric geometry for physicists, see [12, section 3] and of [13, section 4]. Our presentation will follow these references

orthogonal to S_i^u. These are provided by the charges of theory A, R_i^a, courtesy of equation (4.1). These charges then provide a convenient basis of gauge invariant polynomials which parametrise $\mathcal{M}_H^B$,

$$X(k) = \prod_{i=1}^{N} \tilde{q}_i^{\langle R^i,k\rangle} \qquad \text{where } \langle R^i, k\rangle = \sum_{a=1}^{r} R_a^i k_a \,,$$

with $k_a \in \mathbb{Z}^r$. The charges, R_i^a, are now in turn viewed as N vertices in $\mathbb{Z}^r$ and it is these which are used to define Δ as the collection of cones bounded by vectors from the origin through the vertices. Finally, the set F_Δ is defined as containing all sets $\{\tilde{q}_i = 0 \,:\, i \in \{i_\rho\}\,\}$ such that the corresponding set of vertices $\{R_i^a \,:\, i \in \{i_\rho\}\}$ does not lie in any single cone of Δ.

The toric variety defined in this manner has an action of $\mathbf{C}^{\star r}$. For each vertex $R_a^i \in \mathbb{Z}^r$, one may define the action

$$\tilde{q}_j \to \lambda^{\delta_{ij}}\tilde{q}_j \,, \tag{4.2}$$

which is simply the complexified global flavour symmetry of theory B. The limit point as $\lambda \to 0$, of this symmetry is contained $\mathcal{M}_H^B$ (as opposed to F_Δ) as the vertices are themselves the integral generators of one-dimensional cones and therefore clearly belong to a single cone. Notice however that not all linear combinations of the symmetry need necessarily have limit point in $\mathcal{M}_H^B$. Nevertheless, the vertices R_i^a do encode the fixed point structure of all abelian isometries of $\mathcal{M}_H^B$ and allow one to reconstruct the geometry of the toric variety. This is the approach to toric geometry discussed in [11]. To each of the vertices R_i^a, one associates a a hyperplane, D_i, orthogonal to the vertex, on which the corresponding cycle (4.2) is taken to vanish,

$$D_i \equiv \left\{ Y_a \in \mathbb{R}^r \,:\, \sum_{a=1}^{r} R_i^a Y_a = C_i \right\}, \tag{4.3}$$

where C_i are N constants. Define ∇, a polytope of of dimension r, as the region of $\mathbb{R}^r$ bounded by the N hyperplanes D_i. Importantly, by construction, a collection of hyperplanes only intersect if the corresponding limit points are not in F_Δ. This ensures that the Higgs branch, $\mathcal{M}_H^B$ may be viewed as a fibration of $\mathbb{T}^r$ over ∇ such that on the boundary D_i the cycle defined by R_i^a in equation (4.2) shrinks while, on the intersection of k boundaries, k such cycles shrink.

If the vertices of ∇ lie on $\mathbb{Z}^r$, then ∇ is said to be an integral reflexive polytope. These vertices then encode the information for a dual toric variety. This is Batyrev's construction of mirror manifolds [14].

This concludes our discussion of the Higgs branch. The next part of the story is to show that the Coulomb branch of theory A, $\mathcal{M}_C^A$, corresponds to the same toric variety. $\mathcal{M}_C^A$ is parametrised by the r real scalars ϕ_a and r dual photons, σ_a.

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The key point is that the ϕ's are restricted to lie within ∇ due to CS terms, while certain periods of the σ's vanish on the boundaries of this space thus giving a non-trivial fibration of $\mathbb{T}^r$ over ∇ as described above. To see this, take the constants C_i appearing in (4.3) to be defined by $C_i = -m_i$ for $i = 1, \ldots, N$. Then the equation $M_i = 0$, defined in (2.3), specifies the hyperplane D_i, defined in (4.3), spanned by the $\phi_a = Y_a \in \mathbb{R}^r$. Thus the region ∇ is specified by a choice of sign(M_i) for each i. In order to fix this choice, consider the effective CS coupling (2.4),

$$\kappa_{ab}^{\mathrm{eff}} = \kappa_{ab} + \frac{1}{2}\sum_{i=1}^{N} R_i^a R_i^b \,\mathrm{sign}\,(M_i)\,.$$

We see that a judicious choice of bare CS coupling κ_{ab} will ensure that $\kappa_{ab}^{\mathrm{eff}}$ vanishes in ∇ and only in ∇. As long as $\kappa_{ab}^{\mathrm{eff}}$ vanishes, it follows from equation (2.5), that we may choose the bare FI parameters ζ_a in such a way that the effective FI parameters ζ_a^{eff} vanish. In this case the part of the Coulomb branch described by the ϕ_a is precisely ∇.

As the hyperplane D_i is defined by $M_i = 0$, one chiral multiplet becomes massless on each component of the boundary of ∇. Thus, at least in the classical theory, the hyperplane D_i is the root of a Higgs branch on which $q_i \neq 0$ and $q_j = 0$ for all $j \neq i$. As in the simple self-mirror example of the previous section, moving onto this Higgs branch requires changing the FI parameters. In fact, for each i, moving onto the corresponding Higgs branch requires varying the specific linear combination $R_i^a \zeta^a$ away from its critical value. To analyse this transition, it is convenient to promote this linear combination of the FI parameters to a background superfield. After dualizing the gauge fields to periodic scalars, this corresponds to weakly gauging a particular subgroup of the $\mathrm{U}(1)_J^r$ symmetry, $\mathrm{U}(1)_J^{(i)}$, which shifts the corresponding linear combination, $R_i^a \sigma_a$, of the dual photons. Following the same arguments as in section 3, the existence of the new branch in the quantum theory is only consistent if this symmetry is unbroken. At a generic point on the Coulomb branch the whole of $\mathrm{U}(1)_J^r$ is spontaneously broken. Hence, we predict that the subgroup $\mathrm{U}(1)_J^{(i)}$ must be restored on the intersection between the two branches which in turn requires that the corresponding cycle of the toric fibre must degenerate over D_i. One may easily check that this is exactly the same cycle as that defined in equation (4.2). This means that Coulomb branch of theory A precisely agrees with the description of $\mathcal{M}_B^H$ given above as a toric fibration of the polytope ∇. This completes our identification of $\mathcal{M}_C^A$ with $\mathcal{M}_H^B$.

Let us illustrate the above ideas with the simple example of complex projective space. The relevant toric data for this example can also be found in section 4 of [13]. We take,

Theory A: $\mathrm{U}(1)^{N-1}$ gauge theory with N chiral multiplets of charge $R_i^a = \delta_i^a - \delta_i^{a+1}$. The masses are $m_i = -m$ for $i = 1, N$ and $m_i = 0$ for $i = 2, \ldots, N-1$ where $m > 0$.

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The bare CS parameters are

$$\kappa_{ab} = \delta_{ab} - \frac{1}{2}\delta_{a,b-1} - \frac{1}{2}\delta_{a-1,b}\,; \tag{4.4}$$

Theory B: U(1) gauge theory with N chiral multiplets of charge $S_i = +1$ for all i. The masses and CS parameters are set to $-N/2$. The FI parameter is $\tilde{\zeta}$.

It is well known that the classical Higgs branch of theory B is $\mathbb{CP}^{N-1}$ of Kähler class $\tilde{\zeta}$ [10], so we concentrate on the Coulomb branch of theory A. Notice firstly that the charges R_i^a and S_i satisfy (4.1). Following the prescription above, we set $C_i = m$ for $i = 1, N$ and $C_i = 0$ for $i = 2, \ldots, N-1$. The hyperplanes D_i defined in (4.3) are then given by $Y_i - Y_{i-1} = m$, where it is taken that $Y_0 \equiv Y_N \equiv 0$. The vertices of ∇ are given by the intersection points of any $N-1$ of the N hyperplanes, $V_i^a = D_1 \cap \cdots \cap \hat{D}_i \cap \cdots \cap D_N$, where the hat denotes omission of the i^{th} hyperplane. We thus find that $V_i^a = am$ for $i > a$ and $V_i^a = (a-N)m$ for $i \leq a$. Notice that, for $m \in \mathbb{Z}$, all $V_i^a \in \mathbb{Z}$ and ∇ is a reflexive integral polytope as required to define a mirror toric variety. Moreover, as $\sum_{i=1}^N S_i V_i^a = 0$ the vertices of ∇ define another $\mathbb{CP}^{N-1}$. This is simply the statement that the complex projective space is self-mirror. In order to see that the Coulomb branch is defined in terms of ∇, we examine the effective CS coupling,

$$\kappa_{ab}^{\text{eff}} = \kappa_{ab} + \frac{1}{2}\sum_{i=1}^{N} R_i^a R_i^b \,\text{sign}(M_i)\,,$$

where $M_1 = \phi_1 - m$, $M_i = \phi_i - \phi_{i-1}$ for $i = 2, \ldots, N-1$ and $M_N = -\phi_{N-1} - m$. Identifying $Y_a = \phi_a$, we see that the equation for the hyperplane D_i is simply $M_i = 0$ and, with the bare CS coupling given by equation (4.4) to be $\kappa_{ab} = \frac{1}{2}\sum_i R_i^a R_i^b$, we find that $\kappa_{ab}^{\text{eff}} = 0$ for $\text{sign}\, M_i = -1$ or, alternately,

$$-m < \phi_{N-1} < \phi_{N-2} < \cdots < \phi_1 < m\,,$$

which is indeed the polytope ∇. Further, on each boundary $M_i = 0$, the scalar q_i becomes massless and the corresponding cycle arising from shifts in $\sum_a R_i^a \sigma_a$ must shrink. Choosing the ζ_a such that $\zeta_a^{\text{eff}} = 0$, the Coulomb branch is then given as a toric variety to be $\mathbb{CP}^{N-1}$. Notice that in the case of $N = 2$, the above theory differs from the self-mirror theory introduced in the previous section.

Although in the above example the moduli space of vacua is compact, more generally this will not be the case. Indeed, one can check that in the example above the Coulomb branch of theory B is non-compact and coincides with the Higgs branch of theory A. In general, compactness is assured only if the fan Δ spans $\mathbb{Z}^r$. In particular, in the case of a toric variety which obeys the Calabi-Yau condition $\sum_{i=1}^N R_i^u = 0$ for all u, the Coulomb branch is always non-compact.

A. A brane configuration

In this appendix we show how the vacuum structure of the self-mirror theory exhibited in section 3 and, in particular, the compactness of the Coulomb branch can be rederived from a brane realisation of this theory. The relevant brane configuration is the T-dual of that described in [15]. Set-ups identical to the one discussed here were also considered in [16]. We work in units $\alpha' = 1$ and set $g_{\rm st} = 1$.

The configuration of interest involves a D3-brane suspended between various 5-brane webs in IIB string theory. The first of these webs consists of an NS5-brane spanning worldvolume directions 012345, two D5-branes spanning worldvolume directions 012346 and two $(1,1)$ 5-branes in directions 01234(56), where the direction in (X^5, X^6)-plane is $X^5 = \pm X^6$. The final configuration is shown in figure 1 and is positioned at $X^7 = X^8 = X^9 = 0$. The two D5-branes are located at $X^5 = \pm m$ while, when $m = 0$, the NS5-brane is located at $X^6 = 0$. The position of this brane for general m will be described below.

The second 5-brane web is much simpler, consisting of a single NS5-brane spanning 012589, which is traditionally called the NS$'$5-brane. It is positioned at $X^3 = X^4 = 0$, $X^7 = 1/e^2$ and $X^6 = \zeta$. Finally a D3-brane is suspended between the web and NS$'$5-brane, spanning 0127. It is of finite length in the X^7 direction and is positioned at $X^5 = \phi$.

The low-energy dynamics of the D3-brane in this configuration are governed by the three-dimensional gauge theory of section 3, with the dictionary between brane and field theory moduli explicitly given above. Before examining the vacuum structure, we must first describe the zero mode of the 5-brane web corresponding to changing m. This is denoted by the dotted lines in figure 1. Notice that, fixing the position of the $(1,1)$ 5-branes at infinity, the position of the NS5-brane is given by $\zeta = -m$.

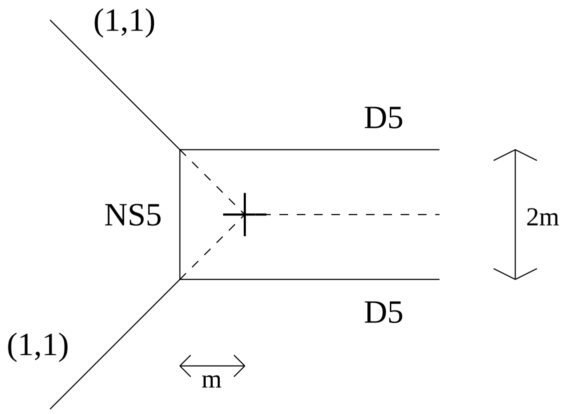

Figure 1: The 5-brane web. The cross denotes the origin of the (X^5, X^6) plane. The dotted lines represent the zero mode of the web.

Let us now discuss the vacuum structure of the theory. The numbering here coincides with that of section 2. Each case is illustrated with a diagram in figure 2 in which we draw only the 5-brane web, with the position of the D3-brane (which also encodes the X^6 position of the NS$'$5-brane) marked by a solid dot.

i) $m = \zeta = 0$: The set up is shown in figure 2i). It is clear that the D3-brane has a unique vacuum state at $\phi = 0$.

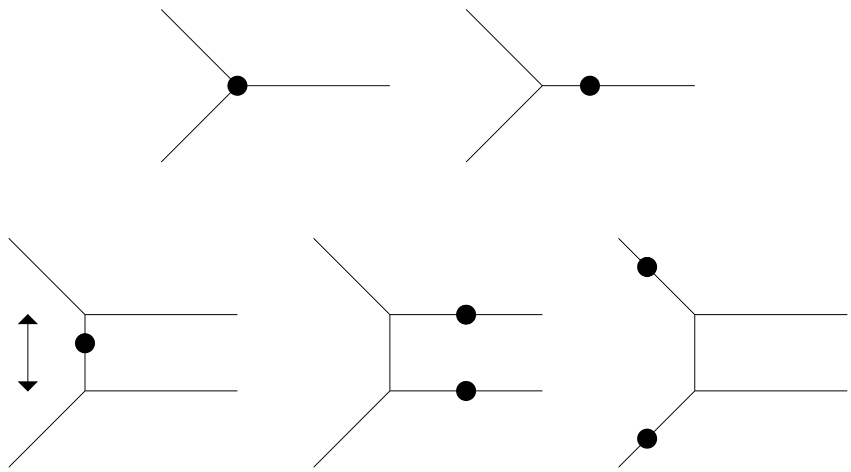

Figure 2: The 5-brane web for various parameters. The dot denotes the position of the D3-brane. The top two diagrams are, from left to right, 2i) and 2ii) and the bottom three 2iii), 2iv) and 2v). In diagram 2iii), the D3-brane is restricted to lie on the interval as shown by the arrow.

ii) $m = 0$, $\zeta > 0$: This is shown in figure 2ii). The D3-brane ends on the two coincident D5-branes located at $\phi = 0$. This is the Higgs branch of the gauge theory which, in the analogous two-dimensional theory, was argued in [15] to contain a copy of $\mathbb{CP}^1$ that is hard to see from the brane picture.

iii) $m > 0$, $\zeta = -m$: It is clear from the brane diagram, shown in figure 2iii), that the D3-brane must end on the NS5-brane and is is therefore restricted to the interval $|\phi| < m$. This is the Coulomb branch of the gauge theory.

iv) $m > 0$, $\zeta > -m$: There are two isolated vacua where the D3-brane ends on one or the other D5-brane as shown in figure 2iv). These are located at $\phi = \pm m$ in agreement with the field theory analysis of section 2.

v) $m \geq 0$, $\zeta < -m$. In this case the D3-brane must end on one of the two dyonic fivebranes as shown in figure 2v). The corresponding vacua are located at $\phi = \pm\zeta$.

Notice that the 5-brane web encodes information about quantum effects of the gauge theory, in particular the contribution to the renormalised FI parameter arising from weakly gauged global symmetries (the last term in (2.5)). Moreover, it was argued in [17, 16] that when the D3-brane is suspended between 5-branes of different kinds CS terms appear on its world-volume theory. This is again in agreement with the field theory dynamics whereby, for instance, in the Coulomb phase described above the D3-brane cannot move outside the interval due to dynamically generated CS terms. In fact, one can further study the theory of section 3 with a bare CS coupling $\kappa = \pm 1$ by replacing the NS$'$5-brane with a $(1,1)'$-5-brane. It is simple to see that, for $\zeta^{\mathrm{eff}} = 0$ and $m \neq 0$, the D3-brane may lie anywhere on one of the $(1,1)$-5-branes of the web, parametrised by the half-line. A short calculation confirms that this is in agreement with the field theory result.

The similarity between the 5-brane webs and toric skeletons was pointed out by Vafa and Leung [11]. In the case of an NS$'$5-brane above, we saw that, on the Coulomb branch, the D3-brane was restricted to lie on the interval of the NS5-brane: this interval is the toric skeleton for $\mathbb{CP}^1$. Similarly, in the case of a $(1,1)'$-5-brane, the D3-brane is constrained to lie on the half-line of the $(1,1)$-5-brane: this is the toric skeleton for the complex plane. In the field theory, using the results of section 3 and [5], one may see that the size of the S^1 is zero at the origin of the half-line while asympotically, where one may trust the one-loop calculation, it grows linearly, in agreement with the proposal that the Coulomb branch is indeed the complex plane.

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